# XVII. *On the aberrations of compound lenses and object-glasses.* *By* J. F. W. Herschel, *Esq. F. R. S. &c.*

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It has not unfrequently of late been made a subject of reproach to mathematicians who have occupied themselves with the theory of the refracting telescope, that the practical benefit derived from their speculations has been by no means commensurate to the expenditure of analytical skill and labour they have called for, and that from all the abstruse researches of Clairaut, Euler, D'Alembert, and other celebrated geometers, nothing hitherto has resulted beyond a mass of complicated formulæ, which, though confessedly exact in theory, have never yet been made the basis of construction for a single good instrument, and remain therefore totally inapplicable, or at least unapplied, in practice. The simplest considerations, indeed, suffice for the correction of that part of the aberration which arises from the different refrangibility of the differently coloured rays; and accordingly, this part of the mathematical theory of refracting telescopes was soon brought to perfection, and has received no important accession since the original invention of the achromatic object-glass. Indeed the theoretical considerations advanced on this part of the subject by Euler and D'Alembert have even had a tendency to retard its advancement, by appearing to establish relations among the relative refractive powers of media on rays of different colours which later experimental researches have exploded.

In the more abstruse and difficult part of the theory of optical instruments which relates to the correction of the spherical aberration, the necessity of an appeal to the powers of algebraic investigation has been all along acknowledged; and as the subject is confessedly within its reach, and presents none of those difficulties which obstruct our progress in the transcendental analysis, but merely such as arise from the involved nature of the equations, and the number of symbols which enter into them, it might have been expected that the appeal would, long ere this, have been successful, the artist have bowed to the dictates, however oracular, of a theory which he was satisfied had its foundation in unerring truth, and the result of their combined labours have been the attainment of all the perfection the telescope is susceptible of. Unhappily, however, this is far from being the case. Investigations, it is true, have been accumulated on each other; formulæ have been deduced, and even tables computed from them; but the investigations, from their dry and laborious nature, and the almost total want of that symmetry which is especially necessary in so complicated a subject, have been studied by few; the formulæ, requiring a more extensive share of algebraical knowledge than can be expected in a practical optician, are thrown aside by him in despair, and the tables hitherto constructed from theory, being founded on data which may never again occur, are worse than useless, serving only to mislead. In consequence, the best and most successful artists are content to work their glasses by trial, or by empirical rules, embodying the result of numerous preceding trials, and which, therefore, have probably some analogy to what would be the final results of theory, if presented in a tangible shape, and accommodated to the peculiarities of their constructions.

The object of the following investigations is to remove or lighten these objections, by presenting first of all, under a general and uniform analysis, the whole theory of the aberrations of spherical surfaces; and in the next place, by furnishing practical results of easy computation to the artist, disentangled from all algebraical complexity, and applicable, by interpolations of the simplest possible kind, to all the ordinary varieties of the materials on which he has to work. In the execution of the former part of this plan, symmetry and simplicity in the disposal of the symbols, is the object chiefly consulted. To attain this, and at the same time avoid circumlocution in the descriptive part of the processes, I have found it necessary to adopt a language somewhat different from that usually employed by optical writers. Instead of speaking of the *focal lengths* of lenses or the *radii* of their surfaces, I speak of their *powers* and *curvatures*, always designating by the former expression, the quotient of unity by the number of parts of any scale which the focal length is equal to; and by the latter, the quotient similarly derived from the radius in question. This mode of expression does no violence to propriety, as the magnifying power of a lens is really inversely proportional to its focal length, and the curvature of a surface is always understood to be reciprocally as its radius; while it gives us the advantage of expressing concisely and naturally, all the most useful propositions in optics. It is certainly simpler (for example) to say, that "the power of any compound lens is the sum of the powers of its separate component lenses," than to express the same thing by saying that "to obtain the focal length of a compound lens, we must divide the product of the focal lengths of its component lenses, by the sum of all the similar products which can be formed by combining them, omitting one in each combination;" or to announce that "the power of a lens is equal to a certain coefficient multiplied by the difference of the curvatures of its surfaces," than to assert that "the focal length is equal to a certain coefficient multiplied by the product of the radii, and divided by their difference." This contraction in language is so convenient, that I hope to see it generally adopted.

The formulæ in the following pages extend no farther than the second term in the developement of the aberration, or that depending on the squares of the semi-apertures. It would have been easy to have carried them to the fourth, and even higher powers; and should object-glasses of *very* great aperture, in comparison with their focal lengths, be ever constructed, it may become necessary; but the dimensions of our present telescopes are far indeed from calling for the immense complexity of algebraic symbols into which this attempt would plunge us; not to speak of the tediousness of the numerical computations, where equations of the tenth and higher degrees are to be resolved. The general value of the aberration for any number of spherical surfaces placed at any finite distances from each other, is assigned by means of an equation of finite differences of the first order. The integration of this presents no difficulty; but I have thought it unnecessary, in the present paper, to pursue it farther in its developement than was required for its application to the theory of thin lenses placed in contact, and especially to that of double object-glasses, reserving the theory of eye-pieces, microscopes, &c. as well as that of thick lenses, for a second communication, should the Society honour this with a place in their Transactions.

The problem of the destruction of the spherical aberration in a double or multiple lens, is well known to be indeterminate, the algebraic conditions requisite for that purpose furnishing but a single equation (at least when the mean rays only are considered). To fix on the best possible condition for limiting the problem, is a matter of considerable delicacy; D'ALEMBERT has proposed, among others (Opusc. Tom. 3, Art. 742), to annihilate the spherical aberration for rays *of all colours*, a refinement which might almost be termed puerile, were it not for the respect due to so great a name.* It has, besides, the inconvenience of leading to equations of the fourth degree. A much better condition, in every point of view, is another proposed by the same profound geometer, in Art. 758, *viz*: the destruction of the aberration for an object situated out of the axis of the telescope; or in other words, the rendering the whole field of view equally perfect *so far as the object-glass is concerned.* But even this is perhaps carrying refinement too far. The difference of the aberrations of an object-glass in and out of the centre of the field, is so small in ordinary telescopes, as to have escaped (so far as my enquiries have gone), the notice of the best practical opticians

* I pass over the construction proposed by D'ALEMBERT, in Art. 746, as having no recommendation but that of avoiding a biquadratic equation; though, it is true, the radii resulting from it might be used.

(and I have consulted many); nor, of course, has any part of their attention been directed to obviate, experimentally, a source of indistinctness they could not perceive to exist. CLAIRAUT in the Memoirs of the Academy for 1757 has computed the radii of a double object-glass from the condition of their touching throughout the whole extent of their interior surfaces; a very desirable thing in practice, and the curvatures which result are very convenient. CLAIRAUT however has employed in his computations, indices of refraction (1.600 and 1.55) higher, especially the latter, than what are now easily met with; and when the average values, such as are likely to occur most frequently, are employed, the construction becomes imaginary for the more dispersive kinds of glass; and within the limits for which it is real, the radii change so rapidly as to render it difficult to interpolate between their calculated values; so that to the artist who is no algebraist this construction loses much of its real advantage.

There remains a condition unaccountably overlooked (so far as my reading has extended) and which the nature of the formulæ of aberration, as given in the following pages, almost forces on our attention; I mean, the destruction of the aberration not only for parallel rays, or when the telescope is directed to celestial objects, but also for rays diverging from a point at any finite distance. The perfection of the telescope, when directed to land objects, seems to require this; and though, in astronomical telescopes, it may appear uncalled for, the construction possesses other advantages of so high an order as to recommend it even there: these are, 1st. the very moderate curvatures required for the surfaces: in this respect it has the advantage of most, if not all, of the constructions hitherto proposed on theoretical grounds. 2dly. That in this construction, the curvatures of the two exterior surfaces of the compound lens of given focal length vary within extremely narrow limits by any variation in either the refractive or dispersive powers at all likely to occur in practice. This remarkable circumstance affords a simple practical rule applicable in all ordinary cases, for calculating the curvatures in any proposed state of the data, and requiring only the use of theorems with which every artist must be familiar; and at all events, rendering it extremely easy to interpolate between calculated values. 3dly. That the two interior surfaces approach, in all cases, so nearly to coincidence, that no considerable practical error can arise from neglecting their difference, and figuring them on tools of equal radii. Indeed, for a ratio of the dispersive powers a little above the average, they are rigorously coincident, and this construction coincides with that of CLAIRAUT above-mentioned; and so nearly is this approach to equality of curvature sustained throughout the whole extent of the function, that even when the ratio of the dispersive powers is so low as 0.75 : 1 (a case almost useless to consider) the difference amounts to less than a 40th part of the curvature of each.

## § I. *General formulæ for the focal distances and aberrations of any combination of spherical surfaces.*

### 1. *Expression for the focal distance of a single spherical surface.*

Let C be the centre of a surface A M (Pl. xix. fig. 1), on which a ray QM proceeding from a point Q in the axis, is incident, and after refraction let it proceed in the direction $Mq$. Draw PM perpendicular to $QAq$, the axis, and put as follows: $\frac{1}{m}=\frac{\text{Sin. Incidence}}{\text{Sin. Refraction}}$ out of the medium QAM into $q$AM or = the relative refractive index of the medium on which the ray is incident.

$y$=PM, the semi-aperture

$D=\frac{1}{QA}$, the reciprocal distance of the radiant point from the surface*

$r=\frac{1}{AC}$, the reciprocal radius, or curvature of the surface

Put also, for brevity, $\frac{AC}{AQ}=\frac{D}{r}=e$; $\frac{PM}{MC}=\text{Sin. ACM}=s$, and we shall have

1st. $QM^2=QC^2+CM^2-2QC\,.\,CM.\cos.ACM$.

2d. $\text{Sin. Incid.}=\text{Sin.}QMC=\frac{QC}{QM}\cdot s$.

$\text{Sin. }CMq=\text{Sin. Refrac.}=m\cdot\frac{QC}{QM}\cdot s$.

3d. Angle $CqM=ACM-CMq$.

$\text{Sin.}CqM=\text{Sin. ACM}\,.\,\cos.CMq-\cos.ACM\,.\,\text{Sin.}CMq$.

4th. $Cq=CM\cdot\frac{\text{Sin. }CMq}{\text{Sin. }CqM}$; $Aq=AC+Cq$.

If we put these expressions into algebraic language, and developing them in powers of $s$, neglect all beyond the cube of that quantity, we find

$$QM=\frac{1}{D}\left\{1+e(1+e)\cdot\frac{s^2}{2}\right\}$$

$$\text{Sin. }CMq=m(1+e)\left\{s-e(1+e)\cdot\frac{s^3}{3}\right\}$$

$$\text{Sin. }CqM=s(1-m-me)+\frac{m(1+e)}{2}\left\{1+e+e^2-m-me\right\}s^3.$$

$$Cq=\frac{1}{r}\cdot\frac{m(1+e)}{1-m-me}\left\{1-\frac{(1-m)(1+e)(m+e+me)}{2(1-m-me)}s^2\right\}$$

* In conformity to the language already explained respecting powers and curvatures, may we not call this the *proximity* of the radiant point?

and finally

$$Aq = \frac{1}{(1-m)r-mer} - \frac{1}{r} \cdot \frac{m(1-m)(1+e)^2(m+e+me)}{2(1-m-me)^2} s^2$$

Let $f$ denote the reciprocal focal distance for central rays, and $f+\Delta f$ the same reciprocal distance for the ray incident at M; then (the aperture being regarded as small in comparison with the focal length), the aberration will be represented by $-\frac{\Delta f}{f^2}$; and if we put for $e$ its value in the foregoing equation, we shall have,

$$f=(1-m)r-m\,D; \qquad (a)$$

$$\Delta f = m(1-m)(r+D)^2\left\{mr+(1+m)D\right\}\times\frac{y^2}{2}; \qquad (b)$$

2. *Theory of the foci of spherical surfaces for central rays.*

Before proceeding to investigate the more complicated cases of the aberrations of several surfaces, we will deduce from the first of these expressions the general equations which determine the place of the focus of central rays after refraction at any number of spherical surfaces; equations we shall have occasion to use hereafter. Let $r_1$, $r_2$, $r_3$, &c. be the curvatures of any number of surfaces $A_1$, $A_2$, $A_3$, &c. which form the common boundaries of the media 0, 1, 2, 3, &c. (fig. 2) and let the relative index of refraction out of the medium 0 into 1 be $\frac{1}{m_1}$, out of 1 into 2, $\frac{1}{m_2}$, and so on; also let $\mu_0$, $\mu_1$, $\mu_2$, &c. be the absolute refractive densities or indices of refraction out of a *vacuum* into these several media, then will

$$\frac{\mu_1}{\mu_0}=\frac{1}{m_1},\ \frac{\mu_2}{\mu_0}=\frac{1}{m_1m_2},\ \frac{\mu_3}{\mu_0}=\frac{1}{m_1m_2m_3}$$

and so on. Moreover, let $t_1$, $t_2$, $t_3$, &c. be the respective intervals between the first and second, the second and third surfaces, and so on; or

$$t_1 = A_1 A_2 ; \; t_2 = A_2 A_3 \text{ \&c.}$$

and let $f_1, f_2$, &c. be the reciprocal distances of the central focus after the 1st, 2d, &c. refraction, from the respective surfaces, or the values of $\frac{1}{A_1 q_1}, \frac{1}{A_2 q_2}$, &c. We here suppose the positive values of $r$ to correspond to surfaces whose convexity is turned towards the original radiant Q (provided its distance be positive) while the positive values of $f$ indicate a situation of the focus $q$ on the opposite side of the surface. With regard to $t$, its values are necessarily positive in cases of refraction, but when $m = -1$, which corresponds to those of reflexion, (which are thus equally included in the present analysis,) $t$ has a negative value.

This premised, if we make $D = o$, or the distance of the radiant point infinite, the focus for parallel and central rays will be assigned by the equation

$$f = (1 - m) \,.\, r$$

Let $\phi$ denote this value of $f$, and $\phi_1 = (1 - m_1)\, r_1$, &c.: then will $\phi_1, \phi_2$, &c. denote the reciprocals of the *principal* focal lengths of the several surfaces, *in situ*, i. e. supposing the adjacent media in each case continued to infinity. We have then in general the equation

$$f = \phi - mD.$$

Suppose now $f$ and $f'$, $m$ and $m'$, $\phi$ and $\phi'$ to represent any two consecutive values of $f$, $m$, and $\phi$ in the series $f_1, f_2, f_3$, &c. &c. Then, since the focus after any refraction becomes the radiant point corresponding to the next, we have

$$\frac{1}{D'} = -\left(\frac{1}{f} - t\right) = -\frac{1 - ft}{f}$$

and the equation $f' = \varphi' - m' D'$ becomes

$$f' = \varphi' + \frac{m'f}{1 - ft}. \qquad (c)$$

This is in fact an equation of differences between the consecutive values of $f$, and the general value may therefore be obtained by integration, or its particular ones deduced in succession from each other, when the integration is impracticable from the values assigned to $m$, $\varphi$, and $t$. The greatest simplification it appears to admit, is its reduction to an equation of the 2d order and first degree, which may be performed by assuming

$$f = \frac{u'}{u} + \frac{1}{t}$$

when the equation will become, after the necessary reductions,

$$0 = u'' - \left(\varphi' - \frac{1}{t'} - \frac{m'}{t'}\right) u' + \frac{m'}{t^2} u$$

It will not be necessary to examine particularly all the integrable cases, or to discuss at present the form of the general value of $u$ or $f$ in terms of $\varphi$, $m$, and $t$. This latter subject is elegantly treated by Lagrange, in a Memoir "Sur la Theorie des Lunettes," in the collection of the Academy of Berlin, (Acad. Berl. 1778), to which we may refer. We need only remark that, whatever be its integral, it must necessarily be of the form

$$\frac{\text{M} - \text{ND}}{\text{O} - \text{PD}}$$

The original distance of the first radiant entering as the arbitrary constant, and being therefore always involved in the same simple algebraic form, whatever be the number and position of the surfaces.

3. Two cases of the equation, however, are worthy of a more particular examination. The first is, when the number of surfaces is infinite, and the intervals separating them infinitely

diminished. In this case the refractive power of the medium varies by insensible gradations; and if we suppose both it and the radii of curvature of the layers of equal density to vary according to a given law, we shall have both $\mu$ and $r$, expressed in functions of the depth to which the ray has penetrated at any moment of its course. This is the case with the crystalline lens of the eye. Dr. BREWSTER's observations have demonstrated that this humour increases very rapidly in density from the circumference to the centre; and to apply our general equation to the evaluation of its focal length, we must proceed as follows. Taking $x$ to represent the depth of any layer whose thickness in the middle is $dx$, the curvatures $r$ and $r'$ of its surfaces will be $r$ and $r+dr$. We have also, $t=dx$, $t'=dx+d^2x$, taking $x$ for the independent variable.

Moreover, since $\frac{\mu}{\mu'}=m'$, we have

$$m'=\frac{\mu}{\mu+d\mu}=1-\frac{d\mu}{\mu}$$

and in consequence, $1-m'=\frac{d\mu}{\mu}$. Hence we obtain

$$\phi'=(1-m')\,r'=\frac{rd\mu}{\mu}$$

neglecting the products of the infinitely small quantities, so that our equation ($c$) becomes (since $f'=f+df$)

$$f+df=\frac{rd\mu}{\mu}+\frac{\left(1-\frac{d\mu}{\mu}\right)\cdot f}{1-fdx}$$

which developed, retaining only terms of the first degree, gives

$$f+df=f+f^2\,dx-f\frac{d\mu}{\mu}+\frac{rdm}{\mu}$$

or simply, putting $\mu=e^v$ where $e=2{\cdot}7182818$, &c.

$$o=df+(f-r)\,dv-f^2\,dx. \qquad (d)$$

The integration of this equation, in which $r$ and $v$ are given functions of $x$, must be performed on the hypothesis that when $x=o$, $f=\left(1-\frac{1}{m}\right)r-\frac{D}{\mu}$, where $\mu$ and $r$ have their initial values. Dr. Young has given a solution of a particular case of this difficult problem in his paper on the Mechanism of the Eye, in the Phil. Trans. for 1801, p. 32.

4. The other case of our equation ($c$) which we proceed to examine, is that, where the surfaces are finite in number, and placed close together, so as to form a compound lens infinitely thin in the middle. In this case we have $t=o$, and the equation becomes simply

$$f'=\phi'+m'f$$

which (putting $f_0=-D$, $\mu_1=\frac{1}{m_1}$, $\mu_2=\frac{1}{m_1 m_2}$), &c. gives at once by integration

$$f\,(\text{or } f_n)=\frac{1}{\mu_n}\left\{\mu_1\phi_1+\mu_2\phi_2+\ldots\mu_n\phi_n-D\right\}; \qquad (e)$$

If, after passing out of a vacuum through any number $n$ of surfaces, the ray emerge again into a vacuum, we have $\mu_n=1$, and

$$f=\mu_1\phi_1+\mu_2\phi_2\ldots\ldots+\mu_n\phi_n-D; \qquad (e')$$

If in this equation we put for $\phi_1$, $\phi_2$, &c. their values in terms of $r_1$, $r_2$, &c., and put

$$\mu_1(1-m_1)=k_1,\ \mu_2(1-m_2)=k_2,\ \&c.$$

we shall obtain

$$f=\frac{1}{\mu_n}\left\{k_1r_1+k_2r_2+\ldots\ldots k_nr_n-D\right\}; \qquad (f)$$

Suppose now the radiant point to be infinitely distant, or $D=o$, then will $f$ become the *principal* reciprocal focal length or power of the system; and calling this F, we get

$$F = \frac{1}{\mu_n}\left\{ k_1 r_1 + k_2 r_2 + \dots k_n r_n \right\}; \qquad (f')$$

$$f = F - \frac{D}{\mu_n} \qquad (f'')$$

or, when the last refraction is made into a *vacuum*,

$$f = F - D \qquad (f''')$$

5. Let us imagine a system of *n lenses*, each consisting of a single medium, placed close together in *vacuo*, and as the ray after traversing each separate lens emerges into a *vacuum*, we have $\mu_2 = \mu_4 = \dots$ &c. $= 1$; $m_1\, m_2 = 1, m_3\, m_4 = 1$, &c. and therefore $k_1 = \mu_1 - 1 = -k_2, k_3 = \mu_3 - 1 = -k_4$, &c. so that our expression for F becomes

$$F = k_1 (r_1 - r_2) + k_3 (r_3 - r_4) + \text{\&c.}$$

or, denoting $\mu_1$, $\mu_3$, &c. simply by $\mu$, $\mu'$, &c.

$$F = (\mu - 1)(r_1 - r_2) + (\mu' - 1)(r_3 - r_4) + \text{\&c.} \qquad (g)$$

In the case of a single lens, this reduces itself to its first term, and calling L, L′, &c. the powers of the several lenses, we have

$$L = (\mu - 1)(r_1 - r_2);\ L' = (\mu' - 1)(r_3 - r_4);\ \text{\&c.} \qquad (h)$$

and finally

$$F = L + L' + L'' + \text{\&c.} \qquad (i)$$

which expresses that *the power of a system of lenses* (placed close together and infinitely thin), *is the sum of the powers of its component lenses.* The powers of concave lenses are here regarded as negative, as well as their focal lengths; while the equation $(f''')$ shows that the sum of the reciprocal distances of the object and its image is equal to the power, or reciprocal focal length, of the system.

6. These propositions are sufficiently well known, and comprise the whole theory of the central foci of infinitely

thin lenses in contact. Let us next examine how these results will be modified by taking into consideration a small, but finite thickness in each of the lenses. To this end we may proceed as follows:

If $U=o$ be any equation of differences involving $f, f'$, and $t$, where $t$ and its values are so small as to permit their powers and products to be neglected; suppose $(f)$ to be the value of $f$ deduced from the equation on the supposition that $t=o$, then in general we may take

$$f=(f)+u$$

where $u$ is a quantity of the same order with $t$. If this be put for $f$ in $U=o$, the equation, by developing, and rejecting the powers and products of $u$ and $t$, will take the form

$$o=V+W.u+X.u'+Y.t$$

V, W, X and Y being functions of $(f)$ and $(f')$, and it is evident that V vanishes by reason of the values assigned to these quantities. There remains then a linear equation of the first order between $u$ and $u'$, which is easily integrated. In the case before us, we have

$$u=f'-\varphi'-\frac{m'f}{1-ft}=o$$

which developed becomes

$$f'=\varphi'+m'f+m'f^2t$$

In this, writing $(f)+u$ for $f$, and retaining only the terms multiplied by $u'$, $u$, and $t$, we get

$$u'-m'u=m'(f)^2t$$

and integrating

$$u=m_2m_3....m_n(f_1)^2t_1+m_3.....m_n(f_2)^2t_2+....$$
$$............+m_n(f_{n-1})^2t_{n-1}.$$

Hence it is easy to conclude, that if we have $n$ *lenses* placed in contact, whose thicknesses are respectively $t, t', t''$, &c., and powers (neglecting their thicknesses) L, L$'$, L$''$, &c. their refractive densities being respectively $\mu, \mu', \mu''$, &c. and the curvatures of their *anterior* surfaces, $r, r', r''$, &c. then will the reciprocal distance of the image from the posterior surface of the last lens be given by the equation.

$$\begin{aligned} f = {} & \mathrm{L} + \mathrm{L}' + \mathrm{L}'' + \&c. - \mathrm{D} \\ & + m \left\{ (\mu - 1) r - \mathrm{D} \right\}^2 t \\ & + m' \left\{ \mathrm{L} + (\mu' - 1) r' - \mathrm{D} \right\}^2 t' \\ & + m'' \left\{ \mathrm{L} + \mathrm{L}' + (\mu'' - 1) r'' - \mathrm{D} \right\}^2 t'' + \&c. \ldots\ldots \quad (j) \end{aligned}$$

continued to as many terms as there are lenses. In this equation $m, m', m''$, &c. are the reciprocals of $\mu, \mu', \mu''$, &c.

*General theory of the aberrations of spherical surfaces for rays incident in the plane of the axis.*

7. Let us next proceed to investigate the spherical aberration of any system of surfaces. Suppose the ray, after passing through the $n^{th}$ surface, to be incident on the $(n+1)^{th}$: its aberration here will arise from two causes; 1st. that after traversing the $n$ preceding surfaces, instead of converging to, or diverging from the focus for central rays, its direction was really to or from a point in the axis, distant from that focus by the total aberration of those $n$ surfaces; and, 2dly that, being incident at a distance from the vertex of the $(n+1)$th surface, a new aberration will be produced here, which (being, as well as the other, of small amount) the principles of the differential calculus allow us to regard as independent of it, and which, being computed separately and added to it, gives the whole aberration of the system of $n+1$ surfaces.

The same is true of the small alterations in the values of $f$ produced by the aberrations. If we denote by $\delta f$ the change in the value of $f$ due to the action of the $n$ preceding surfaces by $\delta' f$, that due to the action of the $(n+1)$th, and by $\Delta f$ the total change arising from their combined action, we shall have

$$\Delta f = \delta f + \delta' f$$

Now, 1st. to investigate the partial alteration $\delta f'$ in the value of $f'$ arising from the total alteration $\Delta f$ in that of $f$, we resume our equation $(c)$; and differentiating its first member according to the characteristic $\delta$ and its second according to $\Delta$, we get ($f$ and $f'$ being the only variables)

$$\delta f' = \frac{m' \Delta f}{(1-ft)^2}.$$

2dly. to discover the partial variation $\delta' f'$ of $f'$, arising immediately from the action of the $(n+1)$th surface, we have, by the equation $(b)$ writing $\delta' f'$ for $\Delta f$, and $m'$, $r'$, $D'$, $y'$ respectively for $m$, $r$, $D$ and $y$,

$$\delta' f' = \frac{y'^2}{2} . m' (m'-1) (r' + D')^2 \left\{ m' r' + (1+m') D' \right\}$$

but, in this case, neglecting the fourth and higher powers of the semi-apertures, it is easy to see that

$$y' = y (1 - ft)$$

and substituting this for $y'$ and $-\frac{f}{1-ft}$ for $D'$, the equation becomes,

$$\delta' f' = \frac{y^2}{2} . m' (1-m') (r' - f - fr't)^2 \left\{ m' r' - \frac{(1+m')f}{1-ft} \right\};$$

so that, uniting the two variations, we find

$$\Delta f' - \frac{m'}{(1-ft)^2} \Delta f =$$

$$= \frac{y^2}{2} . m' (1-m') (r' - f - fr't)^2 . \left\{ m' r' - \frac{(1+m')f}{1-ft} \right\}. \qquad (k)$$

If the surfaces be placed close together, or $t = o$, this becomes

$$\Delta f' - m' \Delta f = \frac{y^2}{2} . m' (1-m') (r' - f)^2 \left\{ m' r' - (1+m') f \right\} \ldots\ldots (l)$$

but when this is not the case, perhaps it will be found more convenient to use one of the following equations

$$\Delta f' - \frac{m'}{(1-ft)^2} \Delta f =$$

$$= \frac{y^2}{2}(1-m')f^2 \cdot \left\{\frac{r'-f'}{f'-\varphi'}\right\}^2 (r'-(1+m')f'); \qquad (m)$$

or,

$$\Delta f' - \frac{m'}{(1-ft)^2} \Delta f =$$

$$= \frac{y^2}{2}(1-m')\left\{r'-f-fr't\right\}^2 . (r'-(1+m')f'); \qquad (m')$$

which are derived from it by eliminating $t$, either wholly or partially from the second member, by the help of equation ($c$). These equations are universally integrable, and suffice to assign the aberration in any proposed combination of spherical surfaces, however placed.

*Theory of the aberrations of infinitely thin lenses placed in contact.*

8. Confining ourselves at present to this branch of our subject, it will easily appear on integrating equation ($l$) that

$$\Delta f = \frac{1}{\mu_n}\left\{\mu_1 Q_1 + \mu_2 Q_2 + \ldots \ldots \mu_n Q_n\right\}$$

where we have

$$Q_n = \frac{y^2}{2} m_n (1-m_n)(r_n - f_{n-1})^2 \left\{m_n r_n - (1+m_n) f_{n-1}\right\}$$

in which it will be recollected that the value of $f_0$ is $-$D.

Let us now examine the composition of this function more particularly; and first, supposing $n=2$, the case of a single lens placed in *vacuo*, we have $\mu_2 = 1$, $m_2 = \frac{1}{m_1} = \mu_1$ and

$$\Delta f = \mu_1 Q_1 + Q_2$$

If then we write $\mu$ and $m$ for $\mu_1$, and $m_1$, and make all reductions, we get

$$Q_1 = \frac{m(1-m)}{2} y^2 \left\{m r_1^3 + (1+3m) r_1^2 D + (2+3m) r_1 D^2 + (1+m) D^3\right\}$$

$$Q_2 = \frac{1-m}{2m^3} y^2 \left\{ \begin{array}{l} r_2^3 + (m^2+2m-3) r_2^2 r_1 + (2m^3 - m^2 - 4m + 3) r_2 r_1^2 - (1+m)(1-m)^3 r_1^3 \\ + \left\{ (3m+m^2) r_2^2 + (4m^3 + 2m^2 - 6m) r_2 r_1 + 3m(1+m)(1-m)^2 r_1^2 \right\} D \\ + \left\{ (2m^3 + 3m^2) r_2 - 3m^2 (1+m)(1-m) r_1 \right\} D^2 + m^3 (1+m) D^3. \end{array} \right\}$$

The expression of $\mu Q_1 + Q_2$ in terms of $r_1$ and $r_2$ will be simplified, if we recollect that when $r_1 = r_2$ the lens will have no aberration, being in this case merely an infinitely thin spherical lamina, equally thick in all parts, through which all rays pass without deviation. $\Delta f$ must consequently vanish when $r_1 = r_2$, and must therefore have $r_1 - r_2$ for a factor; so that $\mu Q_1 + Q_2$ must be divisible by this without remainder. Observing this, we get, on making the reductions,

$$\Delta f = \mu^2 (\mu - 1) \frac{y^2}{2} (r_1 - r_2) \left\{ \begin{array}{l} \alpha r_1^2 + \alpha' r_1 r_2 + \alpha'' r_2^2 \\ + (\beta r_1 + \beta' r_2) D \\ + \gamma D^2 \end{array} \right\}; \quad (n)$$

provided we take

$$\alpha = 2m^3 - 2m + 1; \quad \beta = 4m^3 + 3m^2 - 3m; \quad \gamma = 2m^3 + 3m^2$$
$$\alpha' = m^2 + 2m - 2 \quad \beta' = m^2 + 3m$$
$$\alpha'' = 1$$

or, if we make

$$A = \alpha r_1^2 + \alpha' r_1 r_2 + \alpha'' r_2^2; \quad B = \beta r_1 + \beta' r_2; \quad C = \gamma,$$

then

$$\Delta f = \mu^2 (\mu - 1) \frac{y^2}{2} (r_1 - r_2) \left\{ A + B D + C D^2 \right\}$$

Now, it has been shown that, L being the power of the lens,

$$L = (\mu - 1)(r_1 - r_2)$$

consequently the above value of $\Delta f$ reduces itself to

$$\Delta f = \frac{y^2}{2} \times \mu^2 L \left\{ A + B D + C D^2 \right\} \cdot \qquad (n')$$

9. Suppose now we place any number of lenses close together in *vacuo*, then we shall have, as in Art. 5,

$$\mu_2 = \mu_4 = \&c. = 1\,;\ m_1 m_2 = 1\,;\ m_3 m_4 = 1,\ \&c.$$

But we have also

$$f_3 = \frac{1}{\mu_3}\left\{k_1 r_1 + k_2 r_2 + k_3 r_3 - D\right\}$$

$$= \frac{1}{\mu_3}\left\{k_3 r_3 + f_2\right\}$$

whence we see that $f_3$ is formed from $-f_2$ precisely (*mutatis mutandis*) as $f_1$ is from D; and it is therefore evident that if we take L′, A′, B′, C′, the same functions of the refractive index and radii of the second lens, that L, A, B, C, are of those of the first; and put $D' = -f_2 = -(L - D)$ and write $\mu'$ for $\mu_3$, the refractive index of the second lens, we must have

$$\mu_3 Q_3 + \mu_4 Q_4 = \frac{y^2}{2} \times \mu'^2 L' \left\{A' + B' D' + C' D'^2\right\}$$

and similarly for the value of $\mu_5 Q_5 + \mu_6 Q_6$ adding another accent to the letters in the second member, and observing that $\mu'' = \mu_5$ and

$$D'' = -(L' - D') = -(L + L' - D)$$

and so on, so that we have ultimately, whatever be the number of lenses,

$$\Delta f = \frac{y^2}{2}\left\{\mu^2 L(A + BD + CD^2) + \mu'^2 L'(A' + B'D' + C'D'^2) + \&c.\right\}; \quad (o)$$

continued to as many terms as there are lenses.

If the surfaces of a compound lens be in optical contact, (*i. e.* if the media of which it consists, instead of having thin lenses of air or *vacuum* interposed, be contiguous, the convexity of one fitting exactly into the concavity of the other, as in the case of two glass lenses inclosing a fluid), we may still regard them as separated by infinitely thin, non-refractive laminæ, having equal curvatures on both sides, for it is obvious that these will produce no deviation. In this case, if the curvatures of the proposed system of surfaces be $r_1, r_2, r_3$, &c., those of the equivalent system will be $r_1$ and $r_2$, $r_2$ and $r_3$, &c. In this case we have

$$L = (\mu - 1)(r_1 - r_2)\,;\ L' = (\mu' - 1)(r_2 - r_3) \text{ \&c.}$$

and taking these as the values of L, L′, &c. the equation *(o)* will still hold good.

*Of the mode of correcting the aberrations of a compound lens; and first, of the destruction of the spherical aberration in two lenses of the same medium placed close together, with a view to the improvement of magnifiers, eye-glasses, and burning lenses.*

10. The value of $\Delta f$ in a single lens, for parallel rays, is represented by $\frac{y^2}{2} \cdot \mu^2 L A$. If we put for A its value, and attempt to make this vanish by assigning a relation between $r_1$ and $r_2$, we shall find the roots of the equation imaginary, unless the refractive index exceed 4, a case which nature affords no approach to. If we would reduce it to its minimum value, we find

$$\frac{r_2}{r_1} = \frac{2 - m - 4m^2}{2 + m}.$$

In ordinary glass, we have $\mu = 1.524$, or nearly $\frac{3}{2}$ whence we get $m = \frac{2}{3}$, $\alpha = +\frac{7}{27}$, $\alpha' = -\frac{6}{27}$, $\alpha'' = 1$, $\beta = +\frac{32}{27}$, $\beta' = +\frac{66}{27}$, $\gamma = +\frac{52}{27}$. In such glass therefore we have $\frac{r_2}{r_1} = -\frac{1}{6}$,* or the lens must be double convex or double concave, having the radius of the posterior surface six times that of the anterior.

* In strictness 0.1466, or little more than $\frac{1}{7}$; but this part of the subject being of less moment, I have used $\frac{3}{2}$ for the value of $\mu$, to facilitate the calculations.

The aberration of such a lens being computed (for a given power L) will be found to equal $-\frac{15}{14}y^2$ L. Let this be called $\omega$ and we shall have the proportional aberrations of the following lenses as below:

| | | |
|---|---|---|
| Plano-convex or concave, plane side first | . | $4{\cdot}2 \times \omega$ |
| Do. curved surface first | . . | $1{\cdot}081 \times \omega$ |
| Double-equi-concave or convex | . . . | $1{\cdot}567 \times \omega$ |

The aberration, for parallel rays, of a double lens, of which the first glass has $a$ and $b$ for the curvatures of its surfaces, and the second (of the same substance) $a'$ and $b'$, is represented by

$$-\frac{y^2}{24(L+L')^2}\left\{\begin{array}{l}L(7a^2-6ab+27b^2) \\ +L'(7a'^2-6a'b'+27b'^2-(32a'+66b')L+52L^2)\end{array}\right\}$$

In this if we suppose $a = h\,.\,2\,L$, $a' = h'\,.\,2\,L'$, which give $b = (h-1)\,.\,2\,L$, $b' = (h'-1)\,.\,2\,L'$

$$\frac{b}{a} = \frac{h-1}{h},\ h = \frac{a}{a-b},\ \&c.$$

and suppose moreover, $x = \frac{L'}{L}$, and

$$X = (28\,h'^2 - 48\,h' + 27)\,x^3 + (33 - 49\,h')\,x^2 + 13\,x + (28\,h^2 - 48\,h + 27),$$

we shall have, for the expression of the aberration of the compound lens,

$$-\frac{y^2(L+L')}{6}\cdot\frac{X}{(1+x)^3} = \Omega\,.$$

The aberration of the best single lens of equal power is, as we have already found, $-\frac{15}{14}y^2(L+L')$, and comparing the two, we have

$$\frac{\Omega}{\omega} = \frac{7}{45}\cdot\frac{X}{(1+x)^3}; \qquad (p)$$

11. If we would destroy the aberration, we have only to put $X = o$. As this cubic equation must have at least one

real root, it follows that whatever be the proportion of the curvatures of the surfaces of two thin lenses placed close together, it is always possible to adjust their focal lengths so as to produce a combination free from spherical aberration; and the same is true if the lenses be formed of different materials.

To take an example or two; suppose the first and third surfaces on which the light falls to be plane, and we have $a = a' = o$, and consequently $h = h' = o$, so that the equation $X = o$ becomes

$$27\,x^3 + 33\,x^2 + 13\,x + 27 = o$$

whose only real root is $x = -1\cdot392$. Hence, if we take $L = \pm 1$, we have $L + L' = \mp 0\cdot392$. So that the power of the compound lens is about $\frac{2}{5}$ that of the first glass and of an opposite nature, which, though moderate, may not be too low to be of some use.

If an object be placed in the focus of parallel rays so formed, the rays it sends to every part of the surface will emerge rigorously parallel. Such an object will therefore be seen by an eye on the other side with as much distinctness as if it were a real object at an infinite distance, subtending the same angle, and the combination may thus be used as an eye-glass or magnifier, as well as an object-glass, only reversing its position with respect to the eye.

12. Let $a = b' = o$, $h = o$, $h' = 1$, or suppose the first and last surfaces plane, and our equation $X = o$ becomes

$$7\,x^3 - 16\,x^2 + 13\,x + 27 = o$$

which has but one real root, $x = -0\cdot8517$, or

$$L' = -L \,.\, 0\cdot8517\,;\quad L + L' = L \times 0\cdot1483$$

The lenses then must be of opposite characters, but the power of the compound being only about $\frac{1}{7}$ of that of the first lens, is too low to be of service.

On the same hypothesis as to the plane surfaces, if we trace the variation of the function $\frac{X}{(1+x)^3}$, we shall find that it admits a minimum for a positive value of $x$ given by the equation

$$37x^2 - 58x - 68 = o$$

*viz.* $x = 2{\cdot}349$. This gives

$$L' = 2{\cdot}35 \times L; \; L + L' = 3{\cdot}35 \times L$$

and, $$\frac{\Omega}{\omega} = 0{\cdot}24841$$

This is the minimum value which the ratio of aberrations admits for a positive value of $x$. The combination is represented in Pl. XIX. fig. 3; and we see that a very material superiority over the best single lens of the same power is the result of such a disposition, the aberration being reduced to less than a fourth part. Even if the plano-convex lenses thus laid together be of equal focus, the value of $\frac{\Omega}{\omega}$ will be only 0·6028, indicating still a sensible advantage gained over any single lens.

13. Let us however take up the problem more generally, and enquire what should be the curvatures of all the four surfaces to destroy the whole aberration in the most advantageous manner with respect to the power of the resulting combination. To this end it is evident, that (the equation $X = o$ still subsisting) we must also have $L + L =$ a maximum, and since we may assume as given the power of the first lens (without which the problem is indefinite), we have $dL = o$ and $dL' = o$, whence, $dx = o$ also, differentiating then the equation $X = o$, on this hypothesis, we have

$$(56h-48)dh+\{(56h'-48)\,x-49\}x^2dh'=o.$$

the independent parts of which being made to vanish separately, we find,

$$h=\frac{6}{7},\ h'=\frac{6}{7}+\frac{7}{8x}$$

The former of these determines at once the form of the anterior glass, which must be a double convex or concave of the best form, (the curvatures as 6 : 1) in its best position. The latter being substituted in X=*o* gives

$$x^3-1{\cdot}4\times x^2-1{\cdot}3125\times x+1=o.$$

all whose roots are real, viz:

$$x=-0{\cdot}9798,\ x=+0{\cdot}5609,\ x=+1{\cdot}8193$$

The first of these values gives the worst possible mode of correcting the aberration, the second lens almost exactly neutralizing the first. The second destroys the aberration by the application of a correcting lens whose effect in altering the power is the smallest, while the third is that which affords the greatest possible power. If we execute the numerical computations in the two latter cases we shall find the dimensions as follows:

| | 2d Case. | 3d Case. |
|---|---|---|
| Focal length of the 1st lens | +10·000 | +10·000 |
| Radius of its 1st surface - | + 5·833 | + 5·833 |
| ——— 2d ——— - | —35·000 | —35·000 |
| Focal length of the 2d lens - | +17·829 | + 5·497 |
| Radius of its 1st surface - | + 3·688 | + 2·054 |
| ——— 2d ——— - | + 6·291 | + 8·128 |
| Focal length of the combination | + 6.407 | + 3·474 |

These combinations are represented in Pl. XIX. figs. 4 and 5.

Whether we ought or not to aim at the rigorous destruction of the aberration of rays parallel to the axis, the use to which the lens is to be applied must decide. In a burning glass it is of the highest importance. A slight consideration will suffice to show, that the difference of temperatures produced in the foci of a double convex lens of equal radii, and one of the same focal length but of the best form, must be very considerable. In order to try whether even the latter might not be improved by the shortening of the focus, and the superior concentration of the exterior rays, by applying a correcting lens of one of the forms above calculated, in spite of the loss of heat in passing through a second glass, I procured two lenses to be figured to the radii assigned in the first column of the foregoing table. They were about three inches in aperture, and when combined as above directed, the aberration was almost totally destroyed, and probably would have been so completely, had the index of refraction proper to the glass been employed, instead of that adopted in our calculation for brevity. Their combined effect as a burning lens appeared to me decidedly superior to that of the first lens used alone, and there is therefore good reason to presume that the effect of the other construction which, with the same loss of heat, affords a much greater contraction of the focus would be still better, and I regret not having tried it in preference.

14. In eye-glasses and magnifiers, if we would examine a minute object with much attention, as a small insect, or (when applied to astronomical purposes) if we would scrutinize the appearance of a planet, a lunar mountain, the nucleus of a comet, or a close double star, where extent of field is of less consequence than perfect distinctness in the central point, too much pains cannot be taken in destroying the central aberra-

tion. There is another case in which an aplanatic eye-glass should be employed, viz. in examining the parabolic figure of a speculum, or the perfect adjustment of an object-glass. If the surface of the speculum or object-glass be divided into concentric annuli by diaphragms, covering different parts of it in succession, the rays incident on these, after crossing in their focus will be spread over corresponding annuli on the surface of the eye-glass, and if the distance between the mirror and eye-glass when adjusted to perfect vision, continue the same for all the annuli, we conclude that the figure of the speculum is perfect. It is so however only with respect to that particular eye-glass; and if the aberrations of this be not corrected, all the pains of the artist will only produce a mirror affected with proportional and opposite imperfections. It is true, the use of a very high magnifying power obviates this objection in great measure, by confining the aberration of the eye-glass within a narrower compass; but it is better in theory, and undoubtedly more convenient in practice, to annihilate it altogether. The aberration in the eye appears to me to be entirely out of the question here, but the consideration of that point would lead us away from the present subject.

On the other hand, when a moderately distinct, but extensive field of view is of more consequence than a perfect, but confined one, as in spectacles, reading glasses, magnifiers of moderate power, and eye-glasses for certain astronomical purposes, the correction of the aberration in the centre of the field, may be sacrificed with little inconvenience. By far the best periscopic combination I am acquainted with, consists of a double convex lens of the best form, but placed in its worst position, for the lens next the eye, and a plano-concave whose focal length is to that of the other as 2·6 : 1 or as 13 : 5, placed in contact with its flatter surface and having its concavity towards the object, as in Pl. XIX. fig. 6, for the farthest: yet for destroying the aberration of rays parallel to the axis, nothing can be worse. In fact our formula ($p$) gives for the aberration in this construction

$$\frac{\Omega}{\omega} = 22·302$$

or about 22 times what the best single lens of equal power would give: yet on accidentally combining two such lenses in this manner, I was immediately struck with the remarkable extent of oblique vision,* with the absence of fatigue, on reading some time with a power much beyond that of the natural eye, and with the freedom from colour at the edges of the field, arising from *the opposition of the prismatic refractions* of the two solids, an advantage which a single meniscus does not possess.

*Theory of object-glasses; and first, of the destruction of the chromatic aberration, or the imperfections arising from the different refrangibility of the rays of light.*

15 The perfection of an object-glass requires that parallel or diverging rays of all colours incident on every point of the glass, should converge to one and the same point, and consequently, that we should have $f$ invariable and $\triangle f$ zero, for all the colours of the spectrum. With regard to the latter

* The focal length of the compound lens tried, was 1·84 inch; the field of tolerably distinct vision extended full 40° from the axis, and the forms of objects were distinguishable (the letters of a book might be read) with management, as far as the 75th degree. The lenses used in this combination should be very thin, and the eye applied as close as possible.

condition, we may content ourselves as already remarked, with annihilating $\Delta f$ for the most luminous rays, but the *former must* be satisfied as rigorously as possible. To fix the colour of a ray, we may either fix its position in a spectrum cast by a prism of a standard substance, or the length of its fits of easy transmission and reflexion in *vacuo*. The latter method is on all accounts preferable. The whole difference then between the lengths of the fits of an extreme red and violet ray being taken for unity, let $c$ be difference between those of a ray of any assumed colour and those of the most luminous ray in the spectrum, $c$ being positive for rays nearer the red end of the spectrum, and negative for those nearer the violet. Then in different media, the refractive indices $\mu$, $\mu'$, &c. for that colour will be functions of $c$ of a form depending on the nature of the media, and which perhaps is not the same for any two media in nature. What this form is in any one medium is at present altogether unknown, but in all, we may represent it by

$$\mu + (\mu - 1)\{pc + qc^2 + rc^3 + \&c.\}$$

$p$ being the quantity usually termed the dispersive power of the medium, and which even in the most dispersive bodies hitherto observed does not exceed 0·4, and is generally a very small fraction, while $q$, $r$, &c. are numerical co-efficients, whose influence was perceived shortly after the discovery of the different dispersive powers of bodies, by CLAIRAUT, and whose real existence the experiments of BLAIR, BREWSTER, &c. seem to have placed beyond a doubt. The presumption is that they decrease rapidly in magnitude, and are altogether insensible in the higher terms of the series.

Let this assumed value of the refractive index be put for $\mu$ in our equation

$$f = (\mu - 1)(r_1 - r_2) + (\mu' - 1)(r_3 - r_4) + \&c. - D$$

and similar values for $\mu'$, $\mu''$, &c., and we get

$$\begin{aligned} f = & (\mu-1)(r_1-r_2) + (\mu'-1)(r_3-r_4) + \&c. - D \\ & + c\{p(\mu-1)(r_1-r_2) + p'(\mu'-1)(r_3-r_4) + \&c.\} \\ & + c^2\{q(\mu-1)(r_1-r_2) + q'(\mu'-1)(r_3-r_4) + \&c.\} \\ & + \&c. \end{aligned}$$

or simply (L, L′, &c. designating the powers of the several lenses for the most luminous rays.)*

$$\begin{aligned} f = & L + L' + L'' + \&c. - D; \qquad (q) \\ & + c\{Lp + L'p' + L''p'' + \&c.\} \\ & + c^2\{Lq + L'q' + L''q'' + \&c.\} \\ & + \&c. \end{aligned}$$

In order then that this may remain the same for rays of all colours, it must be verified independent of any particular value assigned to $c$, a condition which gives

$$\left.\begin{aligned} o &= Lp + L'p' + L''p'' + \&c. \\ o &= Lq + L'q' + L''q'' + \&c. \\ & \&c. \end{aligned}\right\}; \qquad (r)$$

These equations, being infinite in number, while the number of the quantities L, L′, &c. is limited, it is of course impossible to satisfy them all, by any adaptation of the latter quantities; but as $q$, $r$, &c. decrease rapidly, we may confine ourselves to satisfy one or two of the first, as the others can produce but

* It were to be wished that in physical optics the *most luminous rays* were always employed as the term of comparison. The mean or middle ray of the spectrum varies in every different medium, and has no distinguishing property which renders it susceptible of exact determination, while the others, by their presence or absence, uniformly mark the maxima and minima of optical phænomena.

an insensible change in the value of $f$, especially since the greatest value of $c$ very little exceeds $\frac{1}{2}$.

Opticians usually regard only the co-efficients $p, p'$, &c. which represent the dispersive powers; and the first of our equations ($r$), which assigns a relation among the powers of the lenses of a very simple nature, has in general been the only one resorted to to insure the achromaticity of the system. It has long however been a subject of complaint, that however perfectly the foci of a double object-glass be adjusted to unite the extreme rays of the spectrum, a more or less considerable quantity of uncorrected colour remains, which cannot be destroyed by such adjustment. This is obviously owing to the non-proportionality of the quantities of $p, q, r$, in different media, which renders it impossible to satisfy more than the first of the equations ($r$), or to what is termed the irrationality of the coloured spaces, in the spectra; and the attention of the optical philosopher has for some time past been turned to the discovery of media, in which either this defect of proportionality shall be imperceptible, or else so considerable, as to admit a more perfect correction by the use of three lenses of different media, so adjusted as to satisfy two of the equations. As the co-efficients $q, r$, &c. furnish equations exactly similar to those afforded by $p$, it would not be amiss, were they designated in future by the epithet of dispersive powers of the 2d, 3d, and superior orders, and were each medium regarded as having its own peculiar system of dispersive powers of all orders to infinity according to the values of $p, q, r$, &c. It is almost superfluous to remark on the very interesting field of experimental enquiry which this view lays open, in which however, little progress can be expected till more rigorous means have been devised of insulating the different homogeneous rays, so as to secure their absolute identity at all times, and under all circumstances, a subject to which I have already devoted some attention, and not altogether without success.

16. In the choice of media, then, for a double object-glass, we must be directed by the condition that their dispersive powers of the first order shall differ considerably. The equation

$$o = \mathrm{L}p + \mathrm{L}'p',$$

the only one we can satisfy rigorously in this case, gives

$$\frac{\mathrm{L}'}{\mathrm{L}} = -\frac{p}{p'}$$

indicating that the lenses must be of opposite characters, and having their focal lengths in the direct ratio of their dispersions. If we call $l$ the power of the compound lens, and take $\varpi = \frac{p}{p'}$, the ratio of the dispersions, we get

$$\mathrm{L} = \frac{l}{1-\varpi}, \quad \mathrm{L} = -\frac{l\varpi}{1-\varpi}$$

We have then only one farther guide to direct us in our choice of media, *viz.* that the dispersive powers of superior orders shall follow as nearly as possible the same proportion in both media, as those of the first.

17. In triple object-glasses we have

$$o = \mathrm{L}p + \mathrm{L}'p' + \mathrm{L}''p''$$

$$o = \mathrm{L}q + \mathrm{L}'q' + \mathrm{L}''q''$$

and $$l = \mathrm{L} + \mathrm{L}' + \mathrm{L}''$$

whence we obtain

$$\mathrm{L} = l \cdot \frac{p'q'' - q'p''}{p(q'-q'') + p'(q''-q) + p''(q-q')}; \qquad (s)$$

$$\mathrm{L}' = l \cdot \frac{p''q - q''p}{p(q'-q'') + p'(q''-q) + p''(q-q')}; \qquad (s')$$

$$\mathrm{L}'' = l \cdot \frac{pq' - qp'}{p(q'-q'') + p'(q''-q) + p''(q-q')}; \qquad (s'')$$

or, $$\frac{L}{L'} = -\frac{\frac{p'}{p''} - \frac{q'}{q''}}{\frac{p}{p''} - \frac{q}{q''}}, \quad \frac{L'}{L''} = -\frac{\frac{p''}{p} - \frac{q''}{q}}{\frac{p'}{p} - \frac{q'}{q}}$$

In order then that this construction should be applicable to any useful purpose, the media must be such as to give moderate values to L, L′, L″, which will (generally speaking) be insured, provided none of the quantities $\frac{p}{p'}$, $\frac{p}{p''}$, $\frac{p'}{p''}$, approach very near in magnitude to the corresponding values $\frac{q}{q'}$, $\frac{q}{q''}$, $\frac{q'}{q''}$, or in other words, provided the media differ considerably in the scales of their dispersive powers.

*Developement and application of the equations for correcting the spherical aberration.*

18. The reciprocal distance of the focus of any combination of lenses or spherical surfaces from the posterior surface, is universally resolvable into a series of the form

$$M + Ny^2 + Oy^4 + Py^6 + \&c.$$

where $y$ is the semi-aperture, or distance of the point of the first surface on which the ray falls, from the axis. In order then that this may be rigorously the same for rays incident on every point of the surface, this must be independent of $y$, and of course we must have

$$N = o, \quad O = o, \quad P = o, \ \&c.$$

In ordinary telescopes however, $y$ is sufficiently small to admit of our neglecting its fourth and higher powers with perfect impunity. Taking the focal length of the telescope for unity, if we allow an inch of aperture to every foot of focal length, we shall have $y = \frac{1}{24}$, $y^4 = \frac{1}{331776}$, &c. So that the remaining terms may be safely neglected in our present enquiry, and I shall accordingly confine myself to the equation $N = o$, or $\Delta f = o$.

If we develope this equation, it will assume the form

$$o = S + T \, . \, D + U \, . \, D^2$$

where S, T, U are functions of the curvatures and powers of the lenses. Now as the telescope may be directed to objects at all different distances, D is arbitrary and independent, and in consequence, the above equation must be satisfied, if possible, independent of D. This gives the three equations $S = o$, $T = o$, $U = o$, or, obtaining the values of these quantities from our equation *(o)*

$$\left.\begin{aligned} o = {} & \mu^2 L A \\ & + \mu'^2 L' \left\{A' - B' L + C' L^2\right\} \\ & + \mu''^2 L'' \left\{A'' - B'' (L + L') + C'' (L + L')^2\right\} \\ & + \&c. \end{aligned}\right\} ; \qquad (t)$$

$$\left.\begin{aligned} o = \mu^2 L B & + \mu'^2 L' \left\{B' - 2 C' L\right\} \\ & + \mu''^2 L'' \left\{B'' - 2 C'' (L + L')\right\} \\ & + \&c. \end{aligned}\right\} ; \qquad (u)$$

$$o = \mu^2 L C + \mu'^2 L' C' + \mu''^2 L'' C'' + \&c. \qquad (v)$$

19. Let us first consider the equation *(t)*. If we put for A, A′, B′, &c. their values in terms of $r_1$, $r_2$, $r_3$, &c. and moreover if we suppose

$$r_1 = r, r_3 = r', r_5 = r'', \&c.$$

and
$$r_1 - r_2 = \varrho, r_3 - r_4 = \varrho', r_5 - r_6 = \varrho'', \&c.$$

$$\mu^2 (\alpha + \alpha' + \alpha'') = a, \; \mu^2 (\alpha' + 2 \alpha'') = b, \; \mu^2 \alpha'' = c$$

$$\mu^2 (\beta + \beta') = e, \; \mu^2 \beta' = f, \; \mu^2 \gamma' = g$$

and similarly for the other lenses, accenting the letters *a*, *b*, *c*, &c. the equation will become

$$o = L\left\{ar^2 - b\varrho r\right\} + L'\left\{a'r'^2 - (b'\varrho' + Le')\,r'\right\}$$
$$+ L''\left\{a''r''^2 - (b''\varrho'' + (L+L')e'')\,r''\right\}$$
$$+ \&c.$$
$$+ Lc\varrho^2 + L'c'\varrho'^2 + L''c''\varrho''^2 + \&c.$$
$$+ L'Lf'\varrho' + L''(L+L')f''\varrho'' + \&c.$$
$$+ L'L^2g' + L''(L+L')^2g'' + \&c.$$

and finally, substituting in this equation for $\varrho$, $\varrho'$, &c. their values $\frac{L}{\mu-1}$, $\frac{L'}{\mu'-1}$, &c., and for $a, b, c, e, f, g$, &c. their values deduced from the equations of Art. 8, we obtain; ($w$)

$$o = L\left\{(2m+1)r^2 - \frac{2\mu+1}{\mu-1}Lr\right\}$$
$$+ L'\left\{(2m'+1)r'^2 - \left((4m'+4)L + \frac{2\mu'+1}{\mu'-1}L'\right)r'\right\}$$
$$+ L''\left\{(2m''+1)r''^2 - \left((4m''+4)(L+L') + \frac{2\mu''+1}{\mu''-1}L''\right)r''\right\}$$
$$+ \&c.$$
$$+ \frac{\mu^2L^3}{(\mu-1)^2} + \frac{\mu'^2L'^3}{(\mu'-1)^2} + \frac{\mu''^2L''^3}{(\mu''-1)^2} + \&c.$$
$$+ \frac{3\mu'+1}{\mu'-1}LL'^2 + \frac{3\mu''+1}{\mu''-1}(L+L')L''^2 + \&c.$$
$$+ (2m'+3)L^2L' + (2m''+3)(L+L')^2L'' + \&c.$$

In this equation it will be observed, the quantities $r$, $r'$, &c. relative to the several lenses are not combined with each other by multiplication, nor do they rise above the second degree. If then we assume, or determine from other conditions the powers L, L, &c. the equation takes a form of great simplicity. Now, as we have already seen, the destruction of the chromatic aberration depends on relations between the focal lengths only, without any regard to the curvatures of the surfaces, and therefore furnishes equations tending to this very point. It is a singular circumstance, and it cannot but be regarded as a very fortunate one, that the introduction of another condition quite independent of the correction of the spherical aberration, and which at first sight seems likely greatly to increase the difficulty of the investigation, should on the contrary tend so remarkably to simplify it.

In general, when the focal lengths are assumed, there will be as many unknown quantities $r, r'$, &c. as there are lenses, and the aberration for parallel rays may therefore be destroyed in a great variety of ways, some more, some less advantageous. If, for example, we limit the figure of one of the lenses in any way (as if we assume it plano-convex or concave,) or assign equal curvatures to both its surfaces, &c. such limitation is equivalent to assigning given values to both its radii, and the terms depending on that lens in equation ($w$) pass into the given part of the equation.

20. Let us next consider the equations ($u$) and ($v$). The latter does not involve the radii of the lenses, but only their powers, being in fact when developed.

$$o = (2m+3)\,L + (2m'+3)\,L' + (2m''+3)\,L'' + \&c.; \quad (x)$$

In a double object-glass, this equation will be incompatible with the equation $o = p\,L + p'\,L'$ expressing the condition of achromaticity, and must of course be sacrificed, the latter being of paramount importance.* It is in fact a very secondary consideration to satisfy this condition in telescopes. In the microscope, however, where D is necessarily a quantity

* Unless such a peculiar adjustment of the media should take place as to render the two conditions identical, which would give

$$\frac{p}{2m+3} = \frac{p'}{2m'+3} \text{ or } \frac{p\mu}{3\mu+2} = \frac{p\,\mu'}{3\mu'+2}.$$

It is a mere matter of curiosity to look for media satisfying this equation. Fluor spar combined with rock crystal comes very near it, but among bodies adapted for object-glasses there are probably none to be found except such as would result from mixtures of different liquids.

of the same order with the powers and curvatures of the glasses, it may be a matter of some moment.

21. The equation ($u$) being developed becomes

$$\left.\begin{array}{l} o=(4m+4)\,\mathrm{L}r+(4m'+4)\,\mathrm{L}'r'+(4m''+4)\,\mathrm{L}''r''+\&c. \\ -\left\{\frac{3\mu+1}{\mu-1}\mathrm{L}^2+\frac{3\mu'+1}{\mu'-1}\mathrm{L}'^2+\frac{3\mu''+1}{\mu''-1}\mathrm{L}''^2+\&c.\right\} \\ -\left\{(4m'+6)\,\mathrm{LL}'+(4m''+6)\,(\mathrm{L}+\mathrm{L}')\,\mathrm{L}''+\&c.\right\} \end{array}\right\}; \quad (y)$$

which being of the first degree, adds nothing to the algebraic difficulty of the problem.

22. Let us apply these results to the case of a double object-glass, and putting, as in Art. 16, $\varpi$ for the ratio of the dispersive powers, and writing for L and L′ their values $\frac{l}{1-\varpi}$ and $-\frac{\varpi l}{1-\varpi}$, the equation $(w)$ becomes,

$$o=(2m+1)r^2-\frac{2\mu+1}{\mu-1}\cdot\frac{1}{1-\varpi}\times lr \qquad (z)$$
$$+\left\{\left(\frac{\mu}{\mu-1}\right)^2-(2m'+3)\varpi+\frac{2\mu'+1}{\mu'-1}\varpi^2-\left(\frac{\mu'}{\mu'-1}\right)^2\varpi^3\right\}\frac{l^2}{(1-\varpi)^2}$$
$$+\left\{(4m'+4)-\frac{2\mu'+1}{\mu'-1}\varpi\right\}\cdot\frac{\varpi}{1-\varpi}lr'-\varpi(2m'+1)r'^2$$

while $(y)$ reduces itself to

$$\left.\begin{array}{l} o=(4m+4)r-(4m'+4)\varpi\,.\,r' \\ -\left\{\frac{3\mu+1}{\mu-1}-(4m+6)\varpi+\frac{3\mu'+1}{\mu'-1}\varpi^2\right\}\frac{l}{1-\varpi} \end{array}\right\}; \qquad (A)$$

To reduce these equations into numbers, we may observe that the value of $\varpi$ is that which varies within the most considerable limits. If we combine the least dispersive flint with the most dispersive crown or plate glass which have yet been observed by DOLLOND, BOSCOVICH, ROBISON, BREWSTER, &c. and *vice versâ*, we shall find 0·51 and 0·782 for the minimum and maximum of this quantity; but it is rare to meet with the extremes. Mr. TULLEY was so good as to communicate to me the highest and lowest dispersions of the two sorts of glass, which had occurred to him in the course of his practice, and calculating on these, I find 0·56915 and 0·65617 for the corresponding values of $\varpi$, so that we may fairly take 0·60 for its average value. With regard to $\mu$ and $\mu'$, their limits are much narrower. In crown and plate glass, we have 1·504 and 1·544 for the extreme indices, while in flint 1·5735 and 1·625 are the lowest and highest I have met with any account of. TULLEY's extremes are 1·5735 and 1·599, and it is said to be very rare at present to meet with flint higher than 1·600. In one specimen only have I observed a greater refraction. We may therefore fix on 1·524 and 1·585 for the mean or average values of $\mu$ and $\mu'$. In order however to embrace a greater range of the function, should we be desirous of interpolating, as well as to provide for the possible discovery of a mode of making flint glass of high dispersive power free from veins, (a thing which it seems very reasonable to *hope*, and which the recent liberality of Government in affording facilities to experiments on a large scale for this express purpose, gives us some ground to *expect*) I have computed the coefficients of our equations $(z)$ and (A) for values of $\varpi$ from 0·50 to 0·75 inclusive, and the results are presented in the following tables.

TABLE I. Coefficients of equation $(z)$, for correcting the spherical aberration for rays parallel to the axis. $\left\{\begin{array}{l}\mu = 1·524 \\ \mu' = 1·585\end{array}\right.$

| | | | | | |
|---|---|---|---|---|---|
| $\varpi = 0·50$ | $2·3123 \times r^2$ | $-15·4505 \times lr$ | $+ 31·4786 \times l^2$ | $+2·9596 \times lr'$ | $-1·1309 \times r'^2 = 0$ |
| 0·55 | 2·3123 | −17·1671 | + 38·8609 | +3·1817 | −1·2440 |
| 0·60 | 2·3123 | −19·3130 | + 49·1100 | +3·3702 | −1·3571 |
| 0·65 | 2·3123 | −22.0720 | + 63·9098 | +3·5107 | −1·4702 |
| 0·70 | 2·3123 | −25·7503 | + 86·4217 | +3·5793 | −1·5833 |
| 0·75 | 2·3123 | −30·9007 | +123·1862 | +3·5326 | −1·6963 |

TABLE 2. Coefficients of equation (A) for correcting the aberration of diverging rays, $\mu$ and $\mu'$ as in Table 1.

| | | | |
|---|---|---|---|
| $\varpi = 0{\cdot}50$ | $6{\cdot}6247 \times r$ | $-17\ 6627 \times l$ | $-3{\cdot}2618 \times r' = 0$ |
| 0·55 | 6 6247 | −19·8254 | −3·5880 |
| 0·60 | 6·6247 | −22 6523 | −3·9142 |
| 0·65 | 6 6247 | −26·4274 | −4·2404 |
| 0·70 | 6·6247 | −31 6245 | −4·5666 |
| 0·75 | 6 6247 | −39 0979 | −4·8928 |

23. If these equations be combined, we shall obtain the dimensions of an object-glass free from aberration, both for celestial and terrestrial objects, provided we restrict our views to objects situated in the prolongation of the axis of the telescope. The arithmetical operations necessary for determining the values of $r$ and $r'$ being executed, we shall find that the resulting quadratics admit real roots, and that in consequence there are two sets of curvatures assignable to the surfaces, which satisfy the algebraic conditions. The values of the first set are however objectionable, as they will be found to correspond to meniscus and concavo-convex forms of the crown and flint lenses respectively, of great curvature, and placed together as in Pl. XIX. fig. 7. Those of the other correspond to moderate curvatures and very convenient forms, as represented in Pl. XIX. fig. 8. The values of $r$, $r'$, in this series, and those of $r_2$ and $r_4$ the curvatures of the posterior surfaces, deduced from them as well as those of L, L', the powers of the lenses are set down in the subjoined table, in which (for simplicity) we have taken $l = 1$.

TABLE 3. Values of $r$, $r'$ &c. deduced from equations ($z$), (A).
$\left\{ \begin{array}{l} \mu = 1\cdot524 \\ \mu' = 1\cdot585 \end{array} \right.$

| $\varpi=$ | $r$ or $r_1=$ | $r_2=$ | $r'$ or $r_3=$ | $r_4=$ | L= | L'= |
|---|---|---|---|---|---|---|
| 0·50 | +1·4818 | −2·3350 | −2·4053 | −0·6959 | +2·0000 | −1·0000 |
| 0·55 | 1·4885 | 2·7524 | 2·7772 | 0·6880 | 2·2222 | 1·2222 |
| 0·60 | 1·4910 | 3·2800 | 3·2637 | 0·6996 | 2·5000 | 1·5000 |
| 0·65 | 1·4855 | 3·9670 | 3·9115 | 0·7369 | 2·8571 | 1·8571 |
| 0·70 | 1·4646 | 4·8967 | 4·8005 | 0·8120 | 3·3333 | 2·3333 |
| 0·75 | 1·4121 | 6·2215 | 6·0790 | 0·9508 | 4·0000 | 3·0000 |

24. These values once obtained, it is easy to calculate the radii and focal lengths of the respective lenses, which I have accordingly set down, for the convenience of those who may be inclined to make trial of this construction, as follows.

TABLE 4. Dimensions of an aplanatic double object glass, indices of refraction 1·524 (crown) and 1·585 (flint). Compound focal length 10·0000.

| Ratio of the dispersive powers. | Radius of the 1st. surface. + | Radius of the. 2d. surface. — | Radius of the 3d. surface. — | Radius of the 4th. surface. — | Focal length of the crown lens. + | Focal length of the flint lens. — |
|---|---|---|---|---|---|---|
| 0·50:1 | 6·7485 | 4·2827 | 4·1575 | 14·3697 | 5·0 | 10·0000 |
| 0·55 | 6·7184 | 3·6332 | 3·6006 | 14·5353 | 4·5 | 8·1818 |
| 0·60 | 6·7069 | 3·0488 | 3·0640 | 14·2937 | 4·0 | 6·6667 |
| 0·65 | 6·7316 | 2·5208 | 2·5565 | 13·5709 | 3·5 | 5·3846 |
| 0·70 | 6·8279 | 2·0422 | 2·0831 | 12·3154 | 3·0 | 4·2858 |
| 0·75 | 7·0816 | 1·6073 | 1·6450 | 10·5186 | 2·5 | 3·3333 |

To reduce these values to those required for any other proposed focal length of the compound lens, a simple proportion is all that is necessary.

25. This table and the preceding afford room for one or two remarks of some moment. And, first, with regard to the

putting together of the lenses, it will be observed that for the lower values of $\varpi$, or the more dispersive varieties of flint-glass, the curvature of the third surface is a very little greater than that of the second, so that the glasses when laid together in their proper position will have a minute interval between them. At a certain value of $\varpi$ between 0·55 and 0·60 ($\varpi \doteq 0{\cdot}58$ nearly) this interval vanishes, and the glasses are in contact over their whole surface. For higher dispersive ratios, if laid close together, they would touch in the vertex. This is regarded as an objection in practice, and justly, (especially when the curvatures of the surfaces in contact differ considerably) as their pressure on each other at the centre must tend to distort their figures, and disturb the uniformity of their density, not to speak of the production of the colours of thin plates, whose effect on vision is more problematic. But in fact, the difference of the curvatures in this construction is so very trifling, as to fall within the limits of practical errors, and therefore, if the separation of the two glasses by a ring of metal (which in a 10 feet object-glass, of 5 inches aperture, even in the very unfavourable case of $\varpi = 0{\cdot}70$ need not exceed 1-400th of an inch in thickness) be deemed unadvisable, it may be neglected, and the glasses ground to the same radius, provided only the necessary alterations are made in the other surfaces to preserve the proper proportion of their focal lengths.

26. With regard to the interpolation of the tables above given for intermediate values of $\varpi$; if we cast our eyes down the second and 5th columns, we cannot but be struck by the very small alteration in the values of $r_1$ and $r_4$, the curvatures of the first and last surfaces throughout the whole *useful*

extent of the table, i. e. as far as $\varpi = 0{\cdot}70$, (beyond which it is very unlikely it should ever extend in practice). In fact, these values have, the one a maximum and the other a minimum between $\varpi = 0{\cdot}55$ and $\varpi = 0{\cdot}60$. The principal variation takes place on the values of $r_2$ and $r_3$, which change rapidly as $\varpi$ increases, the whole stress of the adjustment by which the aberration is corrected being laid on these surfaces. We may take advantage of this fortunate and very remarkable circumstance, and assigning to $r_1$ and $r_4$ constant values, such as to give the least average error, employ them to complete the interior curvatures: thus we may announce it as a practical theorem, which in all probability will be found sufficiently exact for use, that *a double object-glass will be free from aberration, provided the radius of the exterior surface of the crown lens be* 6·720, *and of the flint* 14·20, *the focal length of the combination being* 10·000, *and the radii of the interior surfaces being computed from these data, by the formulæ given in all elementary works on optics, so as to make the focal lengths of the two glasses in the direct ratio of their dispersive powers.*

27. It remains to examine the effect of a change of the values of $\mu$ and $\mu'$ on the curvatures. Now the variations to which these quantities are subject being very trifling, we may neglect their squares and products, and we shall have

$$dr = \frac{dr}{d\mu} d\mu + \frac{dr}{d\mu'} d\mu'$$

where $\frac{dr}{d\mu}$ and $\frac{dr}{d\mu'}$, are constant co-efficients, which are most readily computed by repeating the preceding calculations for values of $\mu$ and $\mu'$ a little differing from those before assumed. And first, with respect to $\frac{dr}{d\mu}$: if we take $\mu = 1{\cdot}504$ and

$\mu' = 1·585$, the resulting curvatures and radii will be as in the two following tables, to which we have also subjoined the values of the coefficients of ($z$) to save the trouble of recomputation, should any other equation beside (A) be thought preferable to use with it.

TABLE 5. Values of $r$, $r'$, &c. $\begin{cases} \mu = 1·504 \\ \mu' = 1·585 \end{cases}$

| $\varpi =$ | $r$ or $r_1 =$ | $r_2 =$ | $r'$ or $r_3 =$ | $r_4 =$ |
|---|---|---|---|---|
| 0·50 | +1·5041 | −2·4642 | −2 5168 | −0·8074 |
| 0·55 | 1·5220 | 2·8871 | 2·8880 | 0·7988 |
| 0·60 | 1·5217 | 3 4385 | 3·3916 | 0·8275 |
| 0·65 | 1·5108 | 4·1582 | 4·0636 | 0·8890 |
| 0·70 | 1 4791 | 5·1346 | 4 9893 | 1·0007 |
| 0·75 | 1·4052 | 6·5313 | 6 3259 | 1·1977 |

and the radii being calculated from these in the same manner as before, will come out as follows:

TABLE 6. Radii of an aplanatic object glass. Focal length = 10·0000. Refractive indices 1·504 and 1·585.

| Ratio of dispersive powers. | Radius of the 1st. surface. + | Radius of the 2d. surface. − | Radius of the 3d. surface. − | Radius of the 4th. surface. − |
|---|---|---|---|---|
| 0·50 | 6·6485 | 4·0581 | 3·9733 | 12·3854 |
| 0·55 | 6·5703 | 3·4637 | 3·4626 | 12·5193 |
| 0·60 | 6·57:6 | 2·9082 | 2·9484 | 12·0839 |
| 0·65 | 6·6190 | 2·4049 | 2·4608 | 11·2481 |
| 0·70 | 6·7609 | 1·9476 | 2·0043 | 9 9927 |
| 0·75 | 7·1164 | 1·5311 | 1·5808 | 8·3491 |

TABLE 7. Coefficients of the equation (z) for correcting the spherical aberration of rays parallel to the axis. $\begin{cases}\mu = 1{\cdot}504 \\ \mu' = 1{\cdot}585\end{cases}$

| $\varpi =$ | $\times r^2$ | $\times lr$ | $\times l^2$ | $\times lr'$ | $\times r'^2 = o$ |
|---|---|---|---|---|---|
| 0·50 | 2·3298 $\times r^2$ | —15·9048 $\times lr$ | + 33·2638 $\times l^2$ | + 2·9596 $\times lr'$ | —1·1309 $\times r'^2 = o$ |
| 0·55 | 2·3298 | —17·6720 | + 41·0647 | + 3·1817 | —1·2440 |
| 0·60 | 2·3298 | —19·8810 | + 51·8989 | + 3·3702 | —1·3571 |
| 0·65 | 2·3298 | —22·7211 | + 67·5529 | + 3·5107 | —1·4702 |
| 0·70 | 2·3298 | —26·5077 | + 91·3803 | + 3·5793 | —1·5833 |
| 0.75 | 2·3298 | —31·8095 | + 130·3267 | + 3·5326 | —1·6963 |

The differences between the numbers in tables 5 and 6, and the corresponding numbers in tables 3 and 4, divided by —0·020 (the value of $d\mu$) give the values of $\frac{dr}{d\mu}$, &c. and $\frac{dR_1}{d\mu}$ &c. calling $R_1$ $R_2$, &c. the radii of the respective surfaces. Thus we find

TABLE 8. Values of $\frac{dr_1}{d\mu}$, $\frac{dr_4}{d\mu}$, $\frac{dR_1}{d\mu}$, $\frac{dR_4}{d\mu}$.

| $\varpi =$ | $\frac{dr_1}{d\mu} =$ | $\frac{dr_4}{d\mu} =$ | $\frac{dR_1}{d\mu} =$ | $\frac{dR_4}{d\mu} =$ |
|---|---|---|---|---|
| 0·50 | —1·115 | + 5·575 | + 5·000 | + 99·215 |
| 0·55 | —1·675 | + 5·540 | + 7·405 | + 100·800 |
| 0·60 | —1·535 | + 6·397 | + 6·675 | + 110·490 |
| 0·65 | —1·265 | + 7·605 | + 5·630 | + 116·140 |
| 0.70 | —0·725 | + 9·435 | + 3·350 | + 116·135 |
| 0·75 | + 0·345 | + 12·345 | —1·740 | + 108·475 |

28. Instituting similar computations for the variation of $\mu'$ from 1·585 to 1·600, we shall obtain the following results.

Table 9. Values of $r_1, r_2, r_3, r_4$, $\begin{cases}\mu = 1{\cdot}524\\ \mu' = 1{\cdot}600\end{cases}$

| $\varpi =$ | $r_1 =$ | $r_2 =$ | $r_3 =$ | $r_4 =$ |
|---|---|---|---|---|
| 0·50 | +1·4830 | −2·3338 | −2·3926 | −0·7259 |
| 0·55 | 1·4888 | 2·7521 | 2·7627 | 0·7257 |
| 0·60 | 1·4898 | 3·2812 | 3·2474 | 0·7438 |
| 0·65 | 1·4814 | 3·9711 | 3·8938 | 0·7986 |
| 0·70 | 1·4546 | 4·9066 | 4·7534 | 0·8945 |
| 0·75 | 1·3953 | 6·2383 | 6·0597 | 1·0597 |

Table 10. Values of the radii $\begin{cases}\mu = 1{\cdot}524\\ \mu' = 1{\cdot}600\end{cases}$

| $\varpi =$ | $R_1 =$ | $R_2 =$ | $R_3 =$ | $R_4 =$ |
|---|---|---|---|---|
| 0·50 | +6·7431 | −4·2849 | −4·1795 | −13·7754 |
| 0·55 | 6·7168 | 3·6336 | 3·6196 | 13·7802 |
| 0·60 | 6·7125 | 3·0477 | 3·0794 | 13·4448 |
| 0·65 | 6·7503 | 2·5182 | 2·5682 | 12·5224 |
| 0·70 | 6·8747 | 2·0381 | 2·0906 | 11·1799 |
| 0·75 | 7·1668 | 1·6030 | 1·6503 | 9·4375 |

Table 11. Coefficients of the equation (z) for correcting the spherical aberration of rays parallel to the axis $\mu = 1{\cdot}524$, $\mu' = 1{\cdot}600$.

| | | | | | |
|---|---|---|---|---|---|
| $\varpi = 0{\cdot}50$ | $2{\cdot}3123 \times r^2$ | $-15{\cdot}4505 \times lr$ | $+\ 31{\cdot}4462 \times l^2$ | $+3{\cdot}0000 \times lr'$ | $-1{\cdot}1250 \times r'^2 = 0$ |
| 0·55 | 2·3123 | −17·1671 | + 38·8263 | +3·2389 | −1·2375 |
| 0·60 | 2·3123 | −19·3130 | + 49·0798 | +3·4500 | −1·3500 |
| 0·65 | 2·3123 | −22·0720 | + 63·8984 | +3·6214 | −1·4625 |
| 0·70 | 2·3123 | −25·7503 | + 86·4584 | +3·7333 | −1·5750 |
| 0·75 | 2·3123 | −30·9007 | +123·3404 | +3·7500 | −1·6875 |

*Fig. 1.*

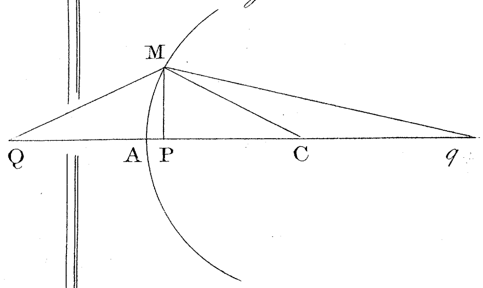



*Fig. 2.*

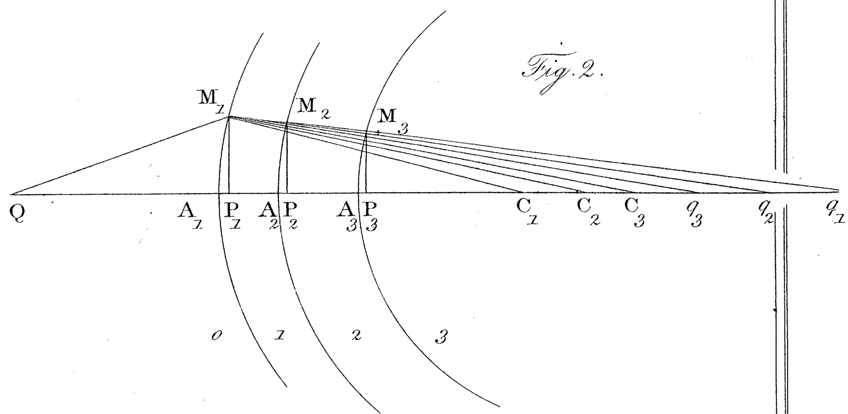



*Fig. 3.*

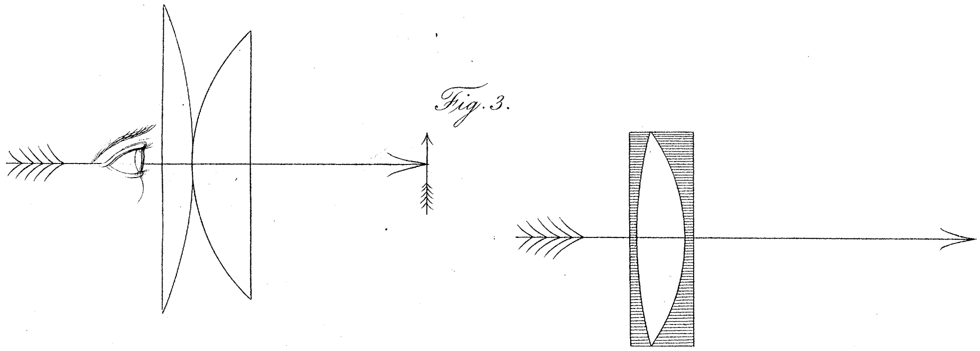


*Fig. 4.*

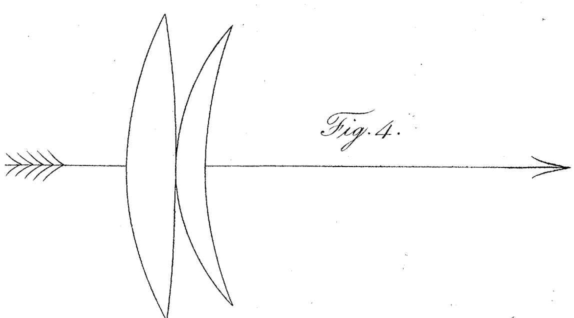


*Fig. 5.*

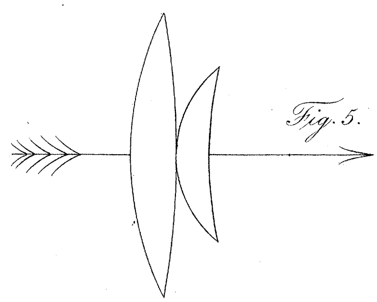


*Fig. 7.*

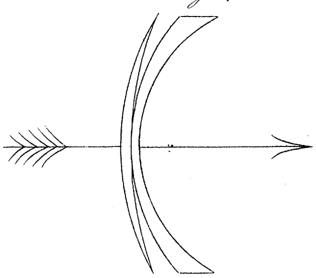


*Fig. 8.*

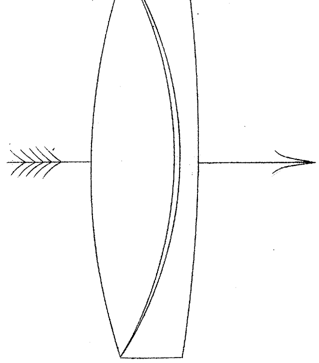
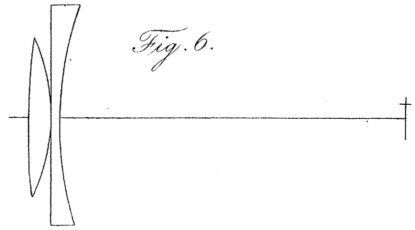


*Fig. 6.*



Js. Basire sc.

TABLE 12. Values of $\frac{dr_1}{d\mu'}, \frac{dr_4}{d\mu'}, \frac{dR_1}{d\mu'}, \frac{dR_4}{d\mu'}$.

| $\varpi =$ | $\frac{dr_1}{d\mu'} =$ | $\frac{dr_4}{d\mu'} =$ | $\frac{dR_1}{d\mu'} =$ | $\frac{dR_4}{d\mu'} =$ |
|---|---|---|---|---|
| 0·50 | +0·080 | −2·000 | −0·360 | +39·620 |
| 0·55 | +0·020 | −2·520 | −0·107 | +50·333 |
| 0·60 | −0·080 | −2·947 | +0·373 | +53· 16 |
| 0·65 | −0·273 | −4·113 | +1·180 | +69·900 |
| 0·70 | −0·667 | −5·500 | +3·133 | +75·700 |
| 0·75 | −1·120 | −7·253 | +5·680 | +72·083 |

It will be seen by this statement, that the variations of the curvatures arising from a variation in the refractive power of the flint lens, are much smaller than those produced by an alteration in that of the crown, which is another fortunate circumstance, the crown and plate glass usually met with being much more uniform in this respect than the flint.

JOHN F. W. HERSCHEL.

*Slough, Feb.* 19, 1821.